

# ANALYSIS OF FREE BOUNDARIES FOR CONVERTIBLE BONDS, WITH A CALL FEATURE

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**ABSTRACT.** Convertible bonds give rise to the so-called free boundary; i.e., an unknown boundary between continuation and conversion regions of the bond. The characteristic feature of such bonds, with an extra call feature, is that its free boundary may reach all the way to the fixed boundary. Our intention in this paper is to study the behavior of the free boundary in the vicinity of a touching point with the fixed boundary. Along the lines of our analysis we will also produce some results on regularity of solutions (value of the bond) up to the fixed boundary.

Our methods are robust and of general nature, and can be applied to fully nonlinear equations. In particular, we shall obtain uniform results for the regularity of both solutions and their free boundaries.

## 1. INTRODUCTION AND BACKGROUNDS

The goal of this paper is to investigate some properties of the solution and the free boundary arising from pricing convertible bonds with the additional call feature, by PDE methods. But first we overview some basic financial notions, we are dealing with, in this paper.

**1.1. What are Convertible Bonds?** *Bond* is a contract which is paid in advance and yields a specified amount on a known date, which is usually called maturity (expiry) date. It is commonly issued by the government or major companies that can guarantee the pay back of the predetermined amount to the holder, on the maturity date. Bonds may also pay a known cash dividend, commonly named coupon, at intervals up to and including the maturity time.

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Face value for a bond (par value) refers to the amount paid to the holder at the maturity time. In this paper we shall take the face value to be  $K$ .

*Convertible Bond* is a bond that entitles the holder the right to convert it into an agreed-upon amount of the company stock or asset, usually at any time of the holder's selection and sometimes at certain times during its lifespan.

*Call feature* refers to the issuing company's right to buy back the bond for an amount  $K$ ; it is by intention we have taken the face value and the call value equal. Clearly, the price of the convertible bond  $V$  is less than this amount, i.e.,  $V(x, t) \leq K$  (otherwise arbitrage to the benefit of the bank will take place.). A convertible bond with call feature is worth less than a standard one.

**1.2. Pricing Convertible Bonds.** Let  $x = x(t)$  be the price of the stock at time  $t$ . Consider a portfolio, consisting of being long one convertible bond and short a number  $\Delta$  of the stock at time  $t$ ,

$$\Pi = V - \Delta x.$$

Suppose that the stock price follows a lognormal random walk,  $dx = rx dt + \sigma x dW$ , where,  $\sigma$  is the volatility of the stock, and  $r$  indicates the interest rate and  $W$  is a Wiener process. Applying Ito's formula, one can derive the changes in portfolio's value

$$d\Pi = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial x} dx + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} dt - \Delta q x dt - \Delta dx + c dt,$$

where  $c$  and  $q$  are the coupon payment of the bond and the dividend of the stock respectively (see [W] for more detailed calculation). Since the return should be at most that of a bank deposit and according to the fact that  $d\Pi = r\Pi dt$ , we conclude

$$r(V - \Delta x) dt \geq \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial x} dx + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} dt - \Delta q x dt - \Delta dx + c dt.$$

In order to have a risk-free portfolio, using  $\Delta$ -hedging we consider  $\Delta = \frac{\partial V}{\partial x}$ . Therefore

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + (rx - q) \frac{\partial V}{\partial x} - rV + c(x, t) \leq 0,$$

with the terminal condition

$$V(x, T) = K.$$

On the other hand, at any time prior to expiry, the bond may be exchanged into  $\gamma$  number of shares, so

$$V \geq \gamma x,$$

where  $\gamma$  is called conversion factor. Furthermore when the stock price moves up, it is apparent that the bondholder's will is to convert the bond into the pre-determined number of shares rather than getting the face value on the expiry date;

$$V \rightarrow \gamma x \quad \text{as} \quad x \rightarrow \infty.$$

Conversely if there is no underlying asset, i.e.,  $x = 0$ , then the bond value satisfies the equation

$$\partial_t V + c = rV,$$

with termination value  $V(0, T) = K$ , and hence admits a solution

$$(1) \quad V(0, t) = Ke^{-r(T-t)} + \frac{c}{r}(1 - e^{-r(T-t)}).$$

**1.3. Notation.** We shall use the following notations in this paper.

$X$ :	$(x, t)$ ,
$K$ :	Call price and Face value of the bond, in this paper both are equal,
$c$ :	Coupon rate (related to the bond),
$q$ :	Dividend rate (related to the stock),
$r$ :	Interest rate,
$\gamma$ :	Conversion factor,
$V(x, t)$ :	Price of the Convertible Bond,
$D_T$ :	$(0, \frac{K}{\gamma}) \times (0, T)$ ,
$B_\rho$ :	$\{y \in \mathbb{R}^n,  x - y  < \rho\}$ ,
$Q_s(X)$ :	$B_\rho \times (t - \rho^2, t + \rho^2)$ ,
$\Lambda$ :	$\{(x, t) : V = \gamma x\}$ ,
$\Gamma$ :	$\partial \{V > \gamma x\} \cap D_T$ (The free boundary),
$\partial_P$ :	Parabolic boundary,
Continuation region :	$\{(x, t) : \gamma x < V(x, t) < K\}$ ,
Call region :	$\{(x, t) : V(x, t) = K\}$ ,
Conversion region :	$\{(x, t) : x \geq 0, V = \gamma x\}$ .

**1.4. Problem Statement.** Let the operator  $\mathcal{L}$  be as follows

$$(2) \quad \mathcal{L} = -\partial_t - \frac{1}{2}\sigma^2 x^2 \partial_{xx} - (r - q)x \partial_x + r.$$

We shall consider the convertible bond with the extra call feature. If  $K$  is the call price, set by the firm, then theoretically we have

$$\gamma x \leq V \leq K.$$

Hence we have the following two constraint variational inequality

$$\begin{cases} \mathcal{L}V = c, & \{\gamma x < V < K\} \cap D_T, \\ \mathcal{L}V \geq c, & \{V = \gamma x\} \cap D_T, \\ \mathcal{L}V \leq c, & \{V = K\} \cap D_T, \\ V(K/\gamma, t) = K, & 0 \leq t \leq T, \\ V(x, T) = K, & 0 \leq x \leq K/\gamma, \end{cases}$$

where  $D_T = (0, \frac{K}{\gamma}) \times (0, T)$ , and  $V \in W_x^{2,2} \cap W_t^{1,2}(D_T)$ . Observe that taking the equation in this sense forces a smooth fit along the free boundary  $\partial\{V > \gamma x\}$ . From a financial point of view it is natural to assume that the value function  $V$  should be continuously differentiable in order to avoid arbitrage. From a pde point of view this falls naturally out when considering variational formulation

$$\min(\mathcal{L}V - c, V - \gamma x) = 0,$$

where the equation is in appropriate sense (weak or viscosity) and with the boundary values as above. It is however not our intention to discuss such aspects of this problem here, but rather pay attention to the qualitative behavior of the free boundary when (and if) it approaches the fixed boundary  $x = K/\gamma$ . From the variational inequality approach—or any other for that matter (Peron's method of smallest supersolution or penalization method)—one always obtains the fact that  $\mathcal{L}V \geq c$  in  $D_T$ . A very good source for study such problems from both Stochastic and Variational point of view is Avner Friedman's book [Fr], Chapter 16.

It should be remarked that in the above setting we did not assign boundary values to the characteristic boundary  $x = 0$ , as the equation is degenerate at such points. Such points are treated by the problem as interior points and one cannot assign boundary data; see [Fich], [OR], and [KN]. Nevertheless the value at  $x = 0$  can be computed directly from the equation, as we did above (see (1))

$$V(0, t) = Ke^{-r(T-t)} + \frac{c}{r}(1 - e^{-r(T-t)}) \quad 0 \leq t \leq T.$$

One can also treat the problem by considering a regularization of the equation in  $\{-K/\gamma < x < K/\gamma, 0 < t < T\}$  where in the set  $\{-K/\gamma < x < 0, 0 < t < T\}$  we consider the evenly reflected coefficients of the operator, which amounts to

$$-\partial_t - \frac{1}{2}\sigma^2(x^2 + \epsilon)\partial_{xx} - (r - q)x\partial_x + r,$$

where  $\epsilon > 0$ .

In this new setting the problem admits a solution (call it  $W_\epsilon$ ) and by uniqueness and symmetry of the problem, the solution is symmetric in  $x$ -variable, and hence

$$(3) \quad \partial_x W_\epsilon(0, t) = 0.$$

From here one may now let  $\epsilon$  tend to zero and we shall obtain a solution to our original problem. This follows by uniform estimates for  $W_\epsilon$  in compact sets of  $D_T$ . On the other hand this convergence is not obvious on the boundary  $\{t = 0\} \cap \partial D_T$ . In particular we *cannot* claim that  $\partial_x V(0, t) = 0$ , at least not without any further analysis. The property (3) will be used in proving that the function  $V$  is monotone increasing in  $x$ -variable but grows slower than  $\gamma x$ . See Proposition 1.1.

Another approach to existence of this type of degenerate problem is the use of a mapping that send the origin to  $-\infty$ , so that the degeneracy appears at infinity point. Then by solving the problem in finite domains, and letting the domain enlarge to the whole space one obtains a solution.

**Assumption 1.1.** *Throughout the paper we shall assume certain conditions to be fulfilled. These conditions force the problem to behave correctly, in a certain sense that is crucial for the problem; e.g. standard maximum principle, uniqueness, compactness should be available. These conditions will in general lead us to existence, uniqueness and qualitative properties which are expected. It should be mentioned that, as we shall discuss it later, these assumptions fall natural for the problem from a financial standpoint. For parabolic PDE we refer to [L] as a general background reference.*

- *Call feature: The call feature as explained above, determines the region  $0 < x < K/\gamma$  where the equations take place.*
- *We shall also assume  $c < rK$ . This assumption forces the value function  $V$  to stay strictly below  $K$  and hence the upper-obstacle never takes place in  $D_T$ . This can, indeed, be seen from equation (4). Since in the set  $\{V = K\} \cap D_T$  we would then have  $\mathcal{L}V = rK \leq c$  contradicting the assumption  $c < rK$ . It should however be remarked that this assumption does not affect the general results in this paper since if  $c \geq rK$  one may consider the obstacle problem with upper obstacle only. When these ingredients vary in  $(x, t)$  and  $c - rK$  changes sign, then the techniques in this paper has to be modified substantially.*
- *We shall further assume that  $c < qK$  otherwise there would be no touch between the graph of  $V$  and that of  $\gamma x$ , i.e. no free boundary. This can be seen from equation (16) below. Indeed*

from (16) we have  $\Lambda \subset \{q\gamma x > 0\}$  so for  $x$  on the free boundary we have this condition as well, and therefore  $x > c/q\gamma$ . On the other hand if  $c > qK$  then  $x > K/\gamma$  and therefore  $x$  is outside the region  $D_T$ .

In most literatures there is an extra assumption  $q \leq r$ , which amounts to the fact that if dividends are higher than interest rate then no body will invest in the bond market.

Since we are just dealing with the case in which  $rK > c$ , when  $V$  is strictly smaller than  $K$ , we may skip the upper obstacle in the formulation of the problem. Also the variational problem produces a supersolution to the PDE and therefore  $\mathcal{L}V \geq c$ , holds in  $D_T$ . We thus look into the following equations of variational inequalities

$$(4) \quad \begin{cases} \mathcal{L}V = c, & D_T \cap \{\gamma x < V < K\}, \\ \mathcal{L}V \geq c, & D_T, \\ V(K/\gamma, t) = K, & 0 \leq t \leq T, \\ V(x, T) = K, & 0 \leq x \leq K/\gamma, \end{cases}$$

with a smooth fit along the free boundary. In other words our solution to the above problem will be  $C_x^{1,\alpha} \cap C_t^\alpha$ , and the corresponding Sobolev spaces are  $W_x^{2,p} \cap W_t^{1,p}$  for  $1 < p < \infty$ .

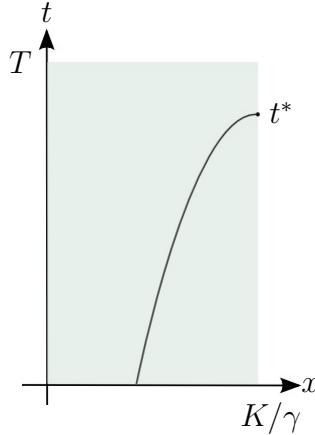


FIGURE 1. The free boundary arising from Problem (1).

An interesting way of thinking is the fact that Black-Scholes formula determines the value of the bond not only for the current and future times, but it can also evaluate the bond in the past, i.e., for  $-\infty < t < T$ .

In the following proposition the existence and the uniqueness of the solution is formulated and then we show the monotonicity and regularity in both  $t$  and  $x$  directions.

**Proposition 1.1.** *There exists a unique solution  $V \in W_{x,loc}^{2,p} \cap W_{t,loc}^{1,p}$ ,  $1 < p < \infty$  to Problem (4) with*

$$(5) \quad 0 \leq V_t \quad 0 \leq V_x \leq \gamma.$$

*In particular the exercise region (if non-empty) is an epi-graph in both  $x$  and  $t$  directions.*

Although the statements of this theorem is well-known to experts, and is considered as "common property", we shall sketch a proof of it in Section 4.

An important feature for convertible bonds is the presence of the free boundary, and the so-called exercise region  $\Lambda$ . If one chooses the ingredients in the problem in a way that conversion is never optimal then this falls under standard theory of bonds. We shall avoid such a discussion here, but we assume from now on that

$$(6) \quad \Lambda \neq \emptyset.$$

Once we agree upon condition (6) then we need to make sure that the touching point between the free boundary and the fixed boundary  $t = t^*$  exists as well as it does not come too close to the termination point  $t = T$ .

**Proposition 1.2.** *In Problem (1), there is  $t^* \in (-\infty, T)$  such that the free boundary  $\Gamma = \partial\{V > \gamma x\}$ , hits the fixed boundary only at  $(K/\gamma, t^*)$  and at no other fixed boundary point.*

To not digress from the discussion, we shall prove this proposition later in Section (4).

**Remark 1.3.** *In Problem (4),  $V_x(K/\gamma, T) = 0$  and  $V_x(K/\gamma, t^*) = \gamma$ . Moreover  $V_x \in C_x^\alpha(\overline{D}_T \setminus \{t = 0\})$  (see Theorem 2.1), hence for every  $x$  there is a  $c_0$ , such that*

$$|V_x(x, t^*) - V_x(x, T)| \leq c_0(T - t^*)^\alpha.$$

*From here we deduce that there is a certain distance between  $t^*$  (in Proposition 1.2) and the maturity date, more exactly*

$$(7) \quad T - t^* \geq \left(\frac{\gamma}{c_0}\right)^{\frac{1}{\alpha}}.$$

## 2. MAIN RESULTS

In this section we shall state our main results concerning qualitative behavior of the solution to Problem (4) and the free boundary arising from it. The proofs are also gathered in the next section.

The following theorem states the optimal smoothness of the solution to Problem (4).

**Theorem 2.1.** *The solution  $V$  to Problem (4) is uniformly  $C_x^{1,1} \cap C_t^{0,1}$  in  $\overline{D}_T \setminus \{t = 0\}$ .*

Finally we state a theorem, which shows that the free boundary is situated outside the set  $\{(x, t), t > -\alpha|x - K/\gamma|^2 + t^*\} \cap D_T$ . In other words, it is located over any arbitrary downward parabola at the point  $(K/\gamma, t^*)$  and after scaling it tends to the line  $x = K/\gamma$ .

**Theorem 2.2.** *The free boundary  $\Gamma = \partial\{V > \gamma x\}$  in Problem (4), is uniformly parabolically tangential to the fixed boundary at  $X^* = (K/\gamma, t^*)$ . In other words there is a modulus of continuity  $\sigma$  ( $\sigma(0^+) = 0$ ) and an  $r_0$  such that*

$$\Gamma \cap Q_{r_0}(X^*) \subset \left\{ (x, t) : t - t^* \leq -\frac{|x - K/\gamma|^2}{\sigma(|x - K/\gamma|)} \right\}.$$

The reader should notice that Theorem 2.2 implies that a parabolic scaling  $(sx, s^2t) + X^*$  of the free boundary at  $X^*$  gives that the limiting free boundary  $\Gamma_0 = \lim_s \Gamma_s$  (where  $\Gamma_s$  is the scaled free boundary) will coincide with the  $t$ -axis.

**Remark 2.3.** *It is notable that the free boundary arising from the American option behaves in a completely different way, i.e., it is located under all arbitrary parabolas and after scaling it tends to the spatial axis.*

**Remark 2.4.** *A final remark concerning Theorem 2.2 is that here we have intentionally avoided to discuss the regularity of the free boundary due to technical reasons. Indeed, one expects the free boundary in this case to be smooth (up to  $C^\infty$ ) but a proof of this "fact" needs detailed analysis and blow-up techniques. There are several papers treating regularity of the free boundary for the american put/call option and it would be likely that these methods apply here, even though not directly. Such a sketch of ideas is presented in [PS]*

## 3. RESTATEMENT OF THE PROBLEM AND TECHNICAL TOOLS

In this section we shall make change of variables so that the main equation falls under general theory of parabolic PDE. We shall also

state some general facts from the standard theory of the free boundary regularity, for parabolic equations (see [PS], [ASU1], and [ASU2]).

**Discussion 3.1.** *For the sake of simplicity first we translate  $V$  to  $\tilde{V}$  by*

$$\tilde{V}(x, t) = V(-x + K/\gamma, T - t).$$

*Inserting this in equation (4), gives*

$$(8) \quad \begin{cases} \tilde{\mathcal{L}}\tilde{V} = c, & D_T \cap \{-\gamma x + K < \tilde{V}\}, \\ \tilde{\mathcal{L}}\tilde{V} \geq c, & D_T, \\ \tilde{V}(0, t) = K, & 0 \leq t \leq T, \\ \tilde{V}(x, T) = K, & 0 \leq x \leq K/\gamma, \end{cases}$$

where  $\tilde{\mathcal{L}}$  is the following operator

$$(9) \quad \tilde{\mathcal{L}} = \partial_t - \frac{1}{2}\sigma^2(-x + K/\gamma)^2\partial_{xx} + (r - q)(-x + K/\gamma)\partial_x + r.$$

One may also derive from (1)

$$(10) \quad \tilde{V}(K/\gamma, t) = Ke^{-rt} + \frac{c}{r}(1 - e^{-rt}), \quad 0 \leq t \leq T,$$

The obstacle should also be transformed and it becomes  $-\gamma x + K$ . Now for

$$u(x, t) = \tilde{V}(x, t) + \gamma x - K,$$

we have

$$\tilde{\mathcal{L}}u = c + (r - q)(-x + K/\gamma)\gamma + r(\gamma x - K) = c - q(-\gamma x + K),$$

in the set  $\{0 < u < \gamma x\}$ . Moreover  $u$  satisfies the following problem

$$(11) \quad \begin{cases} \tilde{\mathcal{L}}u = c - q(-\gamma x + K), & D_T \cap \{0 < u\}, \\ \tilde{\mathcal{L}}u \geq c - q(-\gamma x + K), & D_T, \\ u(0, t) = 0, & 0 \leq t \leq T, \\ u(x, 0) = \gamma x, & 0 \leq x \leq K/\gamma. \end{cases}$$

One also observes that from (10) we have

$$u(K/\gamma, t) = Ke^{-rt} + \frac{c}{r}(1 - e^{-rt}), \quad 0 \leq t \leq T.$$

We also define a scaled operator

$$(12) \quad \tilde{\mathcal{L}}_{s, X^0} := \partial_t - \frac{1}{2}\sigma^2(-(sx + x^0) + K/\gamma)^2\partial_{xx} + s(r - q)(-(sx + x^0) + K/\gamma)\partial_x + s^2r,$$

and its corresponding scaled function

$$v(x, t) = \frac{u(sx + x^0, s^2t + t^0)}{A_s},$$

at the point  $X^0 = (x^0, t^0)$ , for some fixed  $s$ , also in the forthcoming proofs,  $A_s$  will either be  $\sup_{Q_s} u$  or  $s^2$ , dependent on the situation. One can easily verify that

$$(13) \quad \tilde{\mathcal{L}}_{s,X^0} v(x, t) = \frac{s^2}{A_s} (\tilde{\mathcal{L}} u)(sx + x^0, s^2t + t^0).$$

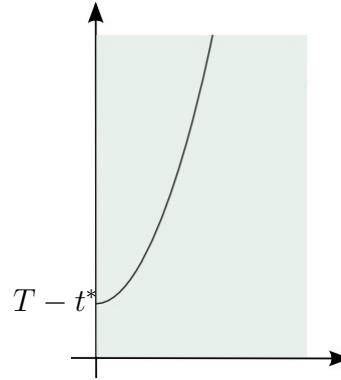


FIGURE 2. The free boundary after transformation

**Lemma 3.2.** *The solution  $u$  to Problem (11) satisfies*

$$(14) \quad \tilde{\mathcal{L}}(u) = (c - q(-\gamma x + K))\chi_{\{u>0\}} \quad \text{in } D_T,$$

along with the boundary conditions

$$(15) \quad u(0, t) = 0 \quad \text{for } 0 \leq t \leq T, \quad u(x, 0) = x\gamma \quad \text{for } 0 \leq x \leq K/\gamma.$$

*Proof.* The proof follows now easily due to smoothness of the function  $u \in W_x^{2,p} \cap W_t^{1,p}$  locally in  $D_T$ , and the equations in (11).  $\square$

**Lemma 3.3.** *(Non-degeneracy for parabolic equations) Let  $u$  be a solution of Problem (14)-(15), then*

$$\sup_{Q_\rho^-(X^0)} u \geq c_0\rho^2 + u(X^0), \quad \text{for } 0 < \rho < K/\gamma,$$

for all  $X^0 \in \overline{\{u > 0\}}$ .

*Proof.* The proof of non-degeneracy is quite standard and it follows by simple maximum principle (see [CPS]). In this particular case with

$u_t = \tilde{V}_t = -V_t \leq 0$  we cannot apply the proof for elliptic case (as is standard for American put/call options); cf. [LS] Lemma 2.2.

We shall present another proof by defining  $h(x, t) = a|x - x^0|^2 - b(t - t^0)$  in the cylinder  $Q_\rho^-(X^0)$ . Let further  $a, b$  be chosen appropriately to have  $\tilde{\mathcal{L}}h \leq c - qK$ , so that  $h$  is a sub-solution to the variational problem locally in  $Q_\rho^-(X^0)$ . Now if  $h \geq u$  on the parabolic boundary  $\partial_p Q_\rho^-(X^0)$  then  $h \geq u$  on  $Q_\rho^-(X^0)$ . In particular if  $X^0 \in \{u > 0\}$  then we arrive at the contradiction  $0 = h(X^0) \geq u(X^0) > 0$ . By continuity we may let  $X^0$  tend to a free boundary point so that the lemma holds for all points in the set  $\overline{\{u > 0\}} \cap D_T$ .  $\square$

The idea of the proof of this lemma appeared in a paper by G.S. Weiss (see Appendix in [We]) for the elliptic case.

#### 4. PROOFS OF THE MAIN RESULTS

**4.1. Proof of Proposition1.1.** We split the proof in cases, according to statement. The proofs being standard, we shall only sketch the ideas. We shall also consider the problem for the translated function  $\tilde{V}$

*Existence:*

To prove the existence of the solution, one can utilize the regularization/penalization or Perron's method, this is standard and follows from various literature, e.g. [F], also see [YY]. Observe that due to degeneracy of the PDE one needs more care in this part, that the usual uniformly elliptic/parabolic case. Therefore an approximation of the problem to a uniformly parabolic problem or alternatively a domain change by a transformation will be needed.

*Partial Regularity:* This as well is quite classical and one can easily find  $C_x^{1,\alpha} \cap C_t^{0,\alpha}$  regularity in the literatures. In general it is regularity in Sobolev spaces that by simple embedding gives such regularity. It should be noticed that once the reformulation of the problem as in Lemma 3.2 is done then one obtains standard parabolic equation with bounded right-hand side. Classical PDE then tells us that the solution is  $C_x^{1,\alpha} \cap C_t^{0,\alpha}$ , for any  $0 < \alpha < 1$ .

*Uniqueness:*

To show the uniqueness of the solution, we suppose that there exist two solutions  $\tilde{V}_1$  and  $\tilde{V}_2$  to Problem (8) and define the set  $\Omega$  as follows

$$\Omega := \{X = (x, t); \tilde{V}_1(X) > \tilde{V}_2(X)\} \subset D_T.$$

We have  $\tilde{V}_1 > \tilde{V}_2 \geq -\gamma x + K$  in  $\Omega$ , hence  $\tilde{\mathcal{L}}(\tilde{V}_1) = c$  in  $\Omega$ . Moreover  $\tilde{\mathcal{L}}(\tilde{V}_2) \geq c$ , i.e.,  $\tilde{V}_2$  is a super-solution to the equation and therefore we can use the comparison principle (see [F] and [L]) to conclude that  $\tilde{V}_2 \geq \tilde{V}_1$  in  $\Omega$ , which leads us to a contradiction. Observe that in the above argument we may use the values at  $x = K/\gamma$ , as both solutions should have the same (non-assigned) value at  $x = K/\gamma$ ; see (10).

*Monotonicity:*

In order to prove the monotonicity in  $t$ -direction we slide the solution  $\tilde{V}$  slightly in  $t$ -direction and obtain  $\tilde{V}_\alpha$

$$\tilde{V}_\alpha(x, t) = \tilde{V}(x, t + \alpha).$$

We want to show  $\tilde{V}_\alpha \leq \tilde{V}$ , in  $D_{T-\alpha}$  for  $\alpha > 0$ .

We have (by inspections)  $\tilde{V}_\alpha \leq \tilde{V}$  on the boundary of  $D_{T-\alpha}$  (where we may also use the value (10)) and therefore according to comparison principle (see the uniqueness part) we can deduce that  $\tilde{V}_\alpha \leq \tilde{V}$  in the domain  $D_{T-\alpha}$ . Observe that the operator does not change, while shifting in  $t$ -direction as the coefficients are independent of  $t$ .

For the monotonicity in  $x$ -direction, a similar slide in  $x$ -direction does not work, since the operator will change. We may does apply the maximum/minimum principle to  $\partial_x W_\epsilon$  (introduced earlier, see (3)) in the continuation (non-coincidence set) for  $W_\epsilon$ .

We first consider  $\tilde{W}_\epsilon = W_\epsilon(-x + K/\gamma, T - t)$  from (3) in the region  $D_T$ , and observe that  $\partial_x \tilde{W}_\epsilon(K/\gamma, t) = \partial_x W(0, t) = 0$ .

Next we apply the maximum principle to the equation  $\partial_x \tilde{\mathcal{L}}(\tilde{W}_\epsilon) = \partial_x c$  in the region  $\{\tilde{W}_\epsilon > -x\gamma + K\}$ , and observe that  $\tilde{V}_x$  satisfies (compare 9)

$$\partial_t(\partial_x \tilde{W}_\epsilon) - \frac{1}{2}\sigma^2(-x + K/\gamma)^2 \partial_{xx}(\partial_x \tilde{W}_\epsilon) + (\sigma^2 + r - q)(-x + K/\gamma) \partial_x(\partial_x \tilde{W}_\epsilon) + q(\partial_x \tilde{W}_\epsilon) = 0,$$

in the region  $\{\tilde{W}_\epsilon > -x\gamma + K\}$ .

On  $\{t = K/\gamma\}$  we have by (3) that  $\partial_x \tilde{W}_\epsilon(K/\gamma, t) = -\partial_x W_\epsilon(0, t) = 0$ , and for  $\partial\{\tilde{W}_\epsilon > -x\gamma + K\}$  as well as for  $\{t = 0\}$  we have  $\partial_x \tilde{W}_\epsilon = -\gamma$ . Finally on  $x = 0$  with  $\{0 < t < T - t^*\}$  we have  $-\gamma \partial_x \tilde{W}_\epsilon \leq 0$ . The last inequality depends on the fact that  $\tilde{W}_\epsilon \geq -x\gamma + K$ .

The maximum and minimum principle both apply and we obtain  $-\gamma \leq \partial_x \tilde{W}_\epsilon \leq 0$ . This naturally shows that the coincidence set (exercise region) for the  $\epsilon$ -problem is an epi-graph. As  $\epsilon$  tends to zero  $W_\epsilon$  tends to  $V$  and we obtain the same properties for  $V$  and its graph.

**4.2. Proof of Proposition 1.2.** We recall from (6) that the  $\Lambda \neq \emptyset$ . By monotonicity of  $V$  in both  $x$  and  $t$ -directions (Proposition 1.1) we

conclude that the set  $\Lambda$  is connected, and its boundary is an epi-graph, in both spatial and time directions. In particular  $\Lambda = \overline{\text{int}(\Lambda)}$  (closure of the interior). Hence  $\partial\Lambda = \partial(\text{int}(\Lambda))$ .

Since  $V = \gamma x$  in the interior of  $\Lambda$  we may differentiate to obtain

$$\partial_t V = \partial_{xx} V = 0 \quad \text{and} \quad \partial_x(\gamma x) = \gamma.$$

Implementing these in equation  $c \leq \mathcal{L}V$  gives  $c \leq -(r - q)x\gamma + r\gamma x$ . Therefore  $c \leq q\gamma x$  for all points in the interior of  $\Lambda$ . Hence there exists a lower bound for  $x \in \Lambda$ , i.e.

$$(16) \quad \Lambda \subset \left\{ x : x \geq \frac{c}{q\gamma} \right\}.$$

This shows that the free boundary can not touch the time axis.

Next, if  $\Lambda$  reaches all the way to  $t = T$ , with  $x < K/\gamma$ , then  $V(x, T) = K$  and in  $\Lambda$  we have  $V(x, T) = \gamma x < K$ , the continuity of  $V$  breaks down, contradicting Proposition 1.1. From both arguments above we conclude that the free boundary neither touches the  $t$ -axis, nor any point on  $t = T$ , with  $x < K/\gamma$ . We need now to exclude the point  $(K/\gamma, T)$ . This is a little bit more tricky and one should use a higher regularity of  $V$  up to the corner points.

It is in general well-known that variational problems produce the same amount of regularity for the solutions as that of the given obstacle in  $x$ -variable and half regularity in  $t$ -variable (but  $C_x^{1,1} \cap C_t^{0,1}$  is in general a regularity threshold). In particular  $\partial_x V$  exists and is continuous up to the boundary, including the point  $(K/\gamma, T)$ . This implies in turn that on one side  $\partial_x V(K/\gamma, T) = 0$  (derivative along the segment  $t = T$ ) and on the other side  $\partial_x V(K/\gamma, T) = \gamma$  (derivative along the segment  $x = K/\gamma$ ). This is a contradiction, and hence there is a  $t^* < T$  such that  $\Lambda \subset \{t < t^*\}$ .

**4.3. Proof of Theorem 2.1.** We first notice that due to (7), (16) and the monotonicity of the  $V$  (Proposition 1.1), the function  $V$  solves an standard parabolic equation in the set

$$\{x < c/q\gamma\} \cup \{t > T - (\gamma/c_0)^{1/\alpha}\}.$$

Hence we can conclude by classical parabolic theory that the  $V$  is as smooth as stated in the theorem, up to the fixed boundary (the ingredients are smooth enough).

Next we will prove the following lemma, which says that if the solutions grows quadratically from the free boundary then one can after rescaling obtain uniform regularity as stated in Theorem 2.1. We state the lemma in terms of the function  $u$ , i.e. a solution to equation (11), or more exactly a solution to (14)- (15).

Let us now introduce some notations:  $\bar{\Gamma} := \Gamma \cup t^*$ ,

$$d^-(X, \partial D_T) = \sup\{\rho : Q_\rho^-(X) \subset D_T\}, \quad d^-(X, \Gamma) = \sup\{\rho : Q_\rho^-(X) \subset D_T \setminus \Lambda\}.$$

**Lemma 4.1.** *Under the assumptions of Theorem 2.1, let  $u$  be a solution of equation (14)- (15). Suppose also that for  $\rho > 0$*

$$(17) \quad \sup_{Q'_\rho(X^0)} u \leq C_0 \rho^2, \quad (Q'_\rho = Q_\rho \cap D_T)$$

for all  $X^0 \in \bar{\Gamma}$ , and a universal  $C_0$  independent of the ingredients in the equations. Then

$$u \in (C_x^{1,1} \cap C_t^{0,1}) (\overline{D_T}),$$

where the norm depends only on the space dimension and the norms of the ingredients.

*Proof.* Consider a point  $X = (x, t) \in D_T \setminus \Lambda$ , set  $s := d^-(X, \Gamma)$ , and let  $Y = (y, \theta) \in Q_1(0)$ . According to the discussion preceding the lemma,  $u$  is regular in  $D_{T-t^*}$ . Hence we may assume  $t \geq T - t^*$  (see figure 2).

Define

$$W(Y) = \frac{u(x + sy, t + s^2\theta)}{s^2}, \quad \text{in } Q_1(0).$$

By 17 we have

$$\|W\|_{L^\infty(Q_1(0))} \leq \sup_{Q_s(X)} \frac{u}{s^2} \leq \sup_{Q_{2s}(\tilde{X})} \frac{u}{s^2} \leq 4C_0.$$

Here  $\tilde{X} \in \Gamma$  is the closest point to  $X$  on the free boundary.

Now  $W$  being bounded in  $Q_1(0)$  and  $\tilde{\mathcal{L}}_{s,X} W = c$ , we can apply interior Schauder estimate, to arrive at

$$|\partial_{xx} W(0, 0)| + |\partial_t W(0, 0)| \leq C.$$

On the other hand  $\partial_{xx} W(0, 0) = \partial_{xx} u(X)$  and  $\partial_t W(0, 0) = \partial_t u(X)$ , which implies

$$u \in (C_x^{1,1} \cap C_t^{0,1}) (\overline{D_T}),$$

as stated in the lemma.  $\square$

To prove Theorem 2.1 we shall only need to show the growth estimate (17) for small  $\rho$ , as the estimate holds for  $d^-(X, \partial D_T)/4 < \rho < d^-(X, \partial D_T)$ . The proof of estimate (17) will follow using the well-known scaling method and blow-up technique, standard in recent theory of free boundary regularity (cf. [ASU1], [ASU2], and [CPS]).

Set

$$S_j(X^0, u) = \sup_{Q'_{2^{-j}(X^0)}} u, \quad Q'_{2^{-j}(X^0)} = Q_{2^{-j}(X^0)} \cap D_T.$$

We claim that for all  $j \in \mathbb{N}$ , and all  $X^0 \in \Gamma$ , and any solution  $u$  to our equation (depending only on the norms of the ingredients )

$$(18) \quad S_{j+1}(X^0, u) \leq \max \{4^{-j}C_1, 4^{-1}S_j(X^0, u), \dots, 4^{-j}S_1(X^0, u)\},$$

for a universal constant  $C_1$ .

Suppose this is true, then we see that (17) follows by inspection. Indeed, for any  $\rho$  (small) we may choose  $j$  such that  $2^{-j-1} \leq \rho < 2^{-j}$ . From (18) then it follows that  $S_\rho \leq S_j$ . Now if the maximum of the right hand side in (18) happens for  $4^{-j}C_1$  then we are done with  $C_0 = 4C_1$ . If not then the maximum is  $4^{-k-1}S_{j-k}$  for some  $k$ , which implies  $S_\rho \leq S_j \leq 4^{-k-1}S_{j-k}$  and we can repeat the argument for  $S_{j-k}$ , until  $k = j$ .

We shall now prove (18), using a contradictory argument. Hence suppose (18) fails. Then, for every  $n \in \mathbb{N}$ , there exist  $X^n \in \bar{\Gamma}$ ,  $u_n$  (solving our problem) and  $j_n \in \mathbb{N}$  such that

$$(19) \quad S_{j_n+1}(X^n, u_n) > \max\{n4^{-j_n}, 4^{-1}S_{j_n}(X^n, u_n), \dots, 4^{-j_n}S_1(X^n, u_n)\}.$$

Since  $S_{j_n+1}(X^n, u_n) > n4^{-j_n}$ , i.e.  $4^{j_n}S_{j_n+1}(X^n, u_n) > n$  we deduce that

$$j_n \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty.$$

Now set

$$R_n = 2^{j_n}(D_T \setminus X^n) = \{X : 2^{j_n}(X - X^n) \in D_T\},$$

and

$$v_n(x, t) := \frac{u_n(2^{-j_n}x + x^n, 2^{-2j_n}t + t^n)}{S_{j_n+1}(X^n, u_n)}, \quad \text{in } R_n,$$

and recall Discussion 3.1. From (12), (13), (14), and (15) we obtain the rescaled equation

$$(20) \quad \tilde{\mathcal{L}}_n v_n(x, t) = \frac{2^{-2j_n}}{S_{j_n+1}} (\tilde{\mathcal{L}} u)(2^{-j_n}x + x^n, 2^{-2j_n}t + t^n) \rightarrow 0,$$

in  $R_n$ . Here  $\tilde{\mathcal{L}}_n := \tilde{\mathcal{L}}_{(2^{-j_n}), X^n}$  (see (12)), and the right hand side tending to zero follows from (14), and (19).

Next the operators  $\tilde{\mathcal{L}}_n$  being uniformly elliptic and  $v_n$  satisfying the equation (20) we may conclude that they are uniformly  $C_x^{1,\alpha} \cap C_t^\alpha$  in any compact subsets of  $R_n$ . In particular by compactness there is a subsequence (relabelled with the same sequence) such that

$$(21) \quad v_n \rightarrow v_0 \quad \text{in } R_\infty \quad v_0 \geq 0 \quad \text{in } R_\infty$$

where  $R_\infty = \lim_n R_n$ , and  $v(0) = 0$ .

On the other hand one can easily deduce from the expression for  $\tilde{\mathcal{L}}_n$  that

$$\tilde{\mathcal{L}}_n \rightarrow \mathcal{L}_0, \quad \text{as } n \rightarrow \infty,$$

where  $\mathcal{L}_0$  is a scaled version of the heat equation, i.e.,

$$\mathcal{L}_0 = \partial_t - \frac{\sigma^2}{2}(-x^0 + K/\gamma)\partial_{xx},$$

and

$$\mathcal{L}_0 v_0 = 0,$$

with  $v(0, 0) = 0$  and  $v_0 \geq 0$ . It is also crucial to see that since  $v_n(0, 0) = 0$  and  $v_n \geq 0$  we have that  $\partial_x v_n(0, 0) = 0$  ( $v_n$  is a  $C_x^1$ ). By the fact that the compactness is in the space  $C_x^{1,\alpha}$  for a  $\alpha > 0$  and this is true up to the boundary of the domain (due to smoothness of the boundary) we conclude that

$$(22) \quad \partial_x v_0(0, 0) = 0$$

Next we see that the domain  $R_\infty$  has three possibilities, depending on the value

$$(23) \quad \lim_j \frac{d^-(X^{j_n}, \partial D_T)}{2^{-j_n}} \in \{0, a_0, \infty\},$$

for some finite number  $a_0 > 0$ . Now in the case  $a_0, \infty$  above we have that the origin is an interior point of the set  $R_\infty$  and  $v_0$  takes a local minimum, which violates the minimum principle for caloric functions (or parabolic pde-s). When the limit in (23) is zero then  $(0, 0) \in \partial R_\infty$  and here one can apply the Hopf boundary point lemma (see [L], Lemma 2.6) to obtain a contradiction to (22).

**4.4. Proof of Theorem 2.2.** For every given  $\epsilon$  define the parabolic-cone  $P_\epsilon$  as follows

$$P_\epsilon = \{X = (x, t); t > \frac{1}{\epsilon}x^2 + (T - t^*)\}.$$

To prove that the free boundary touches the fixed boundary tangentially, we show that for every small  $\epsilon$ , contains the free boundary  $\Gamma$  asymptotically close to the point  $(0, T - t^*)$ . We state and prove this in the following lemma, which leads us to the proof of Theorem 2.2.

**Lemma 4.2.** *Let  $u$  be a solution to our problem. Then for every  $\epsilon > 0$ , there exists  $r_\epsilon$  such that,*

$$\Gamma(u) \cap Q_{r_\epsilon}(0, T - t^*) \subset P_\epsilon \cap Q_{r_\epsilon}(0, T - t^*).$$

*Proof.* Suppose this fails. Then for every  $j \in \mathbb{N}$ , there exist a solution  $u_j$  to our problem and

$$X^j = (x^j, t^j) \in \Gamma(u_j) \cap Q_{r_\epsilon}(0, T - t^*) \setminus P_\epsilon \cap Q_{r_\epsilon}(0, T - t^*),$$

with  $x^j \rightarrow 0$ , and  $t^j \rightarrow T - t^*$ . Let  $s_j = |X^j - (0, T - t^*)|$  (the parabolic distance), and define  $\tilde{u}_j$  be the scaled version of  $u_j$  at  $(0, T - t^*)$  which is defined in the set

$$R_j := \{(x, t) : 0 < x < K/s_j\gamma, -(T - t^*)/s_j < t < t^*/s_j\}.$$

More exactly

$$\tilde{u}_j = \frac{u_j(s_j x, s_j^2 t + T - t^*)}{s_j^2}.$$

Now for every scaled function  $\tilde{u}_j$ , one can find a point  $\tilde{X}^j = (x^j/s_j, t^j/s_j^2) \in \Gamma(\tilde{u}_j)$  with  $|\tilde{X}^j| = 1$ . Next we use Theorem 2.1 to see that  $\tilde{u}_j \in C_x^{1,1} \cap C_t^{0,1}$  locally on each compact set of  $R_j$ . Therefore for some subsequence

$$\tilde{u}_j \rightarrow u_0 \quad \text{and} \quad \tilde{X}^j \rightarrow X^0$$

were  $u_0$  is a global solution in  $R_\infty = \{x > 0\}$ ,  $X^0 \in \Gamma(u_0)$  with  $|X^0| = 1$ .

The scaled operator can also be written as  $\tilde{\mathcal{L}}_j = \tilde{\mathcal{L}}_{s_j, (0, t^*)}$ , which in turn gives the equation

$$\tilde{\mathcal{L}}_j \tilde{u}_j = (c - q(-\gamma s_j x + K)) \chi_{\{\tilde{u}_j > 0\}}.$$

As  $j \rightarrow \infty$  we arrive at a global solution

$$\tilde{\mathcal{L}}_0 \tilde{u}_j = (c - qK) \chi_{\{u_0 > 0\}},$$

with

$$\tilde{\mathcal{L}}_0 = \partial_t - \frac{\sigma^2 K}{2\gamma} \partial_{xx},$$

and  $0 \leq u_0(X) \leq C|X|^2$ ,  $X^0 \in \Gamma(u_0)$ . Rewriting this we have

$$\frac{\sigma^2 K}{2\gamma} \partial_{xx} u_0 - \partial_t u_0 = (qK - c) \chi_{\{u_0 > 0\}},$$

which requires  $qK - c > 0$  for presence of a free boundary; otherwise non-degeneracy cannot be applied.

Here one may scale the operator to reduce it to the heat equation, and then use Theorem II in [ASU2] to claim that  $\partial_t u_0 = 0$  (i.e.  $u_0$  is time independent). But then  $u_0$  is a one dimensional solution to  $D_{xx} u_0 = A \chi_{\{u_0 > 0\}}$  for some  $A > 0$ , and that (by non-degeneracy, Lemma 3.3) both origin and  $x^0$  (in  $X^0 = (x^0, t^0)$ ) are free boundary points. This is a contradiction as simple computations show that  $u_0(x) = Ax_+^2/2$  if the origin is a free boundary point.  $\square$

The parabolic tangentiality of Theorem (2.2) follows from Lemma (4.2), by taking the inverse of the relation  $\epsilon \rightarrow r_\epsilon$ , and denoting it  $\sigma(r)$ .

## 5. DISCUSSION

In closing we want to bring to the readers attention several facts and clarifications.

We have assumed that  $K, r, c$  and many other ingredients in this paper are constants, which actually is not necessary for our main theorems about optimal regularity or the parabolically tangential behavior of the free boundary to work out. All one needs is uniform behavior (in their norm) to have similar results.

The standing conditions  $rK > c$  or/and  $qK > c$  may also be dropped but then a double obstacle may occur and if these value also change in time and stock-value  $x$  then one may have too complicated situations as the bond value  $V$  may touch both upper and lower obstacle and switch between them too many times. Such an analysis requires more deeper insight into the problem. It is noteworthy that double obstacle problems are not so well studied close to a free boundary when both obstacles hit. In our case this is the point  $(K/\gamma, t^*)$ .

Other aspects that may be subject for study, by our methods can be the case of convertible bonds with random interest and call feature. The problem now will become two space-dimensional and more delicate. It is however likely (if not apparent) that our methods work in such cases as well and give similar results.

A final remark that we would want to make is the case when the underlying asset is a combination of several assets, and max/min value for such stocks may be considered as conversion possibility.

We hope to come back to such problems for detailed analysis in the future.

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